JOURNAL OF MATERIALS SCIENCE 21 (1986) 4475-4480

The evaluation of the strength distribution of silicon carbide and alumina fibres by a multi-modal Weibull distribution

KOICHI GODA, HIDEHARU FUKUNAGA
Faculty of Engineering, Hiroshima University, Saijo, Higashi-Hiroshima 724, Japan

The strengh distributions of silicon carbide and alumina fibres have been evaluated by a multi-modal Weibull distribution function. This treatment is based on the concept that the fracture of the fibre is determined by competition among the strength distributions of several kinds of the defect sub-population. Since those fibres were observed to have two types of fracture mode, the evaluation of a bi-modal Weibull distribution was performed in comparison with the single Weibull distribution usually employed. The accuracy of the fit for these two distributions was judged from maximum logarithm likelihoods and cumulative distribution curves. The result showed that the logarithm likelihood calculated using the bi-modal Weibull distribution function gave a larger value, as compared with those using the single Weibull distribution function. The curve predicted from the former function was also in good agreement with the data points. In addition, the strength distribution and the average value at a different gauge length were extrapolated from the Weibull parameters estimated at the original gauge length. In this case, also, the bi-modal Weibull distribution gave a more accurate prediction of the data points.

1. Introduction

Most statistical strength analysis of advanced ceramic fibres, such as boron (e.g. [1, 2]), carbon (e.g. [3-5]), coreless silicon carbide [6], alumina [7], carbon-core silicon carbide [8] fibres, etc., has been discussed using a single Weibull distribution function. This distribution is based on the theory in which the fracture is controlled by the weakest defect of all the defects in a fibre, the so-called "weakest link theory". It has also been reported, on the other hand, that those fibres had several kinds of fracture modes [1-9] and each of the strengths was completely changed depending on the kind of defect causing the failure. We consider, in this case, it is too rough to approximate the strength distribution as a straight line on a Weibull probability graph without taking the kind of the defect into account. In fact it is also reported that a cumulative distribution curve predicted from the single Weibull distribution is inconsistent with experimental data [10]. In recent years, therefore, other methods, e.g. a method using a mixed Weibull distribution function [11] and a multi-stage tension testing procedure based on Poisson's model [12] etc., have been proposed for evaluating the strength distribution of ceramic fibres. ceramic fibres.

In this paper we try to apply a multi-modal Weibull distribution function [13] to the strength distribution of ceramic fibres, from a viewpoint that the distribution is determined by competition among the strength distributions of several kinds of the defect sub-population existing in a fibre. Two types of advanced metal-reinforcing fibres, i.e. silicon carbide and alumina fibres, were employed in the analysis of the present work. It is known that these two types of fibres have two kinds of fracture mode [6, 7, 9]. The Weibull parameters of single and bi-modal Weibull distributions were estimated from the results of Nunes [7] and the present experiment, based on a maximum likelihood estimation method [13, 14]. The accuracy of these two distribution functions was evaluated by the maximum logarithms of likelihoods and the cumulative distribution curves. The result showed that the likelihood calculated from the bi-modal Weibull distribution gave a larger value than that using the single Weibull distribution. The distribution curve predicted from the bi-modal Weibull distribution was also in better agreement with the experimental data. The extrapolative ability of the distribution curve and the average strength was additionally investigated. It was also proved that these values estimated from the bi-modal Weibull distribution corresponded more closely with the experimental data than that from the single Weibull distribution.

2. Analytical procedure

2.1. Description of distribution function

For discussion of the strength distribution of brittle materials, such as ceramics, the following two terms are generally assumed (e.g. [11, 13]):

1. The material contains inherently many strength-limiting defects, and its strength depends on the weakest defect of all of them.
2. There are not interactions among the defects.

0022-2461/86 $03.00 + .12

The present analysis is also carried out on these assumptions. The cumulative distribution function $F(\sigma)$ of strength σ for simple tension, as shown in many reports [6–8, 15], is expressed by the following single Weibull distribution function on the basis of the single-risk model (referred as the case of two-parameter), whether the kind of strength-limiting defect exists a lot in a fibre or not:

$$F(\sigma) = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right] \qquad (1)$$

where m is the shape parameter and σ_0 is the scale parameter. In recent years, however, it was proposed [9] that the distribution should be given by the multi-modal Weibull distribution function based on the multi-risk model [14], if two or more kinds of strength-limiting defect sub-population existed together in a brittle material. Here for simplicity a bi-modal Weibull distribution function is described as follows [13, 15–17]:

$$\begin{aligned} F(\sigma) &= 1 - [1 - F_1(\sigma)][1 - F_2(\sigma)] \\ &= 1 - \exp\left[-\left(\frac{\sigma}{\sigma_{01}}\right)^{m_1} - \left(\frac{\sigma}{\sigma_{02}}\right)^{m_2}\right] \end{aligned} \qquad (2)$$

$F_1(\sigma)$ and $F_2(\sigma)$ mean the strength distribution functions of the defect sub-population Nos. 1 and 2, respectively. Each of them is described as the single Weibull distribution.

For estimating Weibull parameters, a maximum-likelihood estimation method [13] is applied in the present work. The likelihood function of the single Weibull distribution is generally given as

$$L = \prod_{i=1}^{n} f(\sigma_i) \qquad (3)$$

where $f(\sigma)$ means a probability density function and n is the total number of samples. The parameters maximizing L are equivalent to maximum likelihood estimators. On the other hand, the multi-modal Weibull distribution is based on a multi-risk model, so the likelihood function is given by the following equation (referred to as the bi-modal distribution) [13, 14]:

$$\begin{aligned} L &= C\prod_{i=1}^{n_1} f_1(\sigma_i^{(1)})[1 - F_2(\sigma_i^{(1)})]\prod_{j=1}^{n_2} f_2(\sigma_j^{(2)})[1 - F_1(\sigma_j^{(2)})] \\ &= C\left\{\prod_{i=1}^{n_1} f_1(\sigma_i^{(1)}) \prod_{j=1}^{n_2} [1 - F_1(\sigma_j^{(2)})]\right\} \\ &\quad \times \left\{\prod_{j=1}^{n_2} f_2(\sigma_j^{(2)}) \prod_{i=1}^{n_1} [1 - F_2(\sigma_i^{(1)})]\right\} \\ &= CL_1L_2 \end{aligned} \qquad (4)$$

where C is a constant $(= n!/(n_1!n_2!))$, n_1 is the number of samples fractured by the cause of Defect 1 at stress $\sigma_1^{(1)}, \ldots \sigma_{n_1}^{(1)}$ and n_2 is the number of samples fractured by the cause of Defect 2 at stress $\sigma_1^{(2)}, \ldots, \sigma_{n_2}^{(2)}$. Since L_1 and L_2 contain only Weibull parameters of the distribution described as Defects 1 and 2, respectively, the method of finding the maximum likelihood estimates for each likelihood, L_1 and L_2, may be applied (hereinafter referred to as the "two-step maximum likelihood estimation method" [13]). It is also reported that such a method gives more reasonable parameters than the methods of hazard plotting estimation and mean order ranking estimation [17]. Then, the likelihood equations used are as follows, e.g. in the case of L_1

$$\begin{aligned} \frac{1}{m_1} + \frac{1}{n_1}\sum_{i=1}^{n_1} \ln \sigma_i - \frac{\sum_{i=1}^{n} \sigma_i^{m_1} \ln \sigma_i}{\sum_{i=1}^{n} \sigma_i^{m_1}} &= 0 \\ \sigma_{01} &= \left(\frac{1}{n_1}\sum_{i=1}^{n} \sigma_i^{m_1}\right)^{1/m_1} \end{aligned} \qquad (5)$$

As to these equations, the first equation is solved by an iterative technique (the Newton–Raphson method) and if m_1 is found, σ_{01} is easily calculated from the second equation. The values of likelihood are obtained by a substitution of these Weibull estimators for Equations 3 and 4, respectively. It may be evaluated that a distribution function having a larger value of the likelihood gives a more realizable distribution.

2.2. Prediction of strength distribution and the mean value at different gauge lengths

In the case of our predicting a strength distribution at a different gauge length, Equations 1 and 2 are extended as follows:

$$F(\sigma) = 1 - \exp\left[-\frac{L}{L_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right] \qquad (6)$$

$$F(\sigma) = 1 - \exp\left\{-\frac{L}{L_0}\left[\left(\frac{\sigma}{\sigma_{01}}\right)^{m_1} + \left(\frac{\sigma}{\sigma_{02}}\right)^{m_2}\right]\right\} \qquad (7)$$

where L is the length of a specimen and L_0 is the standard length of the specimen at which Weibull parameters have been estimated. In this paper, we deal only with the case of simple tension, so a hazard function $(\sigma/\sigma_0)^m$ is simply proportional to a relative length L/L_0.

An average strength $\bar{\sigma}$ can be calulated using Equations 6 and 7. The average value of Equation 6 is generally given as

$$\bar{\sigma} = \sigma_0\left(\frac{L}{L_0}\right)^{-1/m} \Gamma\left(1 + \frac{1}{m}\right) \qquad (8)$$

where Γ means the gamma function. On the other hand, the average value of Equation 7 has been given by integrating numerically the following equation, since it cannot be expressed as an analytical one:

$$\bar{\sigma} = \int_0^\infty \sigma f(\sigma)\, d\sigma = \int_0^\infty [1 - F(\sigma)]\, d\sigma \qquad (9)$$

3. Experimental procedure and results

3.1. Test materials and method of tensile test

The following two types of advanced metal-reinforcing ceramic fibres were used as the test materials: coreless silicon carbide fibres (Nicalon, produced by Nippon Carbon Co. Ltd hereinafter referred to as "SiC fibres") and alumina fibres (Fiber FP, produced by Du Pont, hereinafter referred to as "Al_2O_3 fibres").

The specimens of the SiC and Al_2O_3 fibres were prepared with a 10 mm gauge length on the basis of Japan Industry of Standards R 7601 (Testing Methods

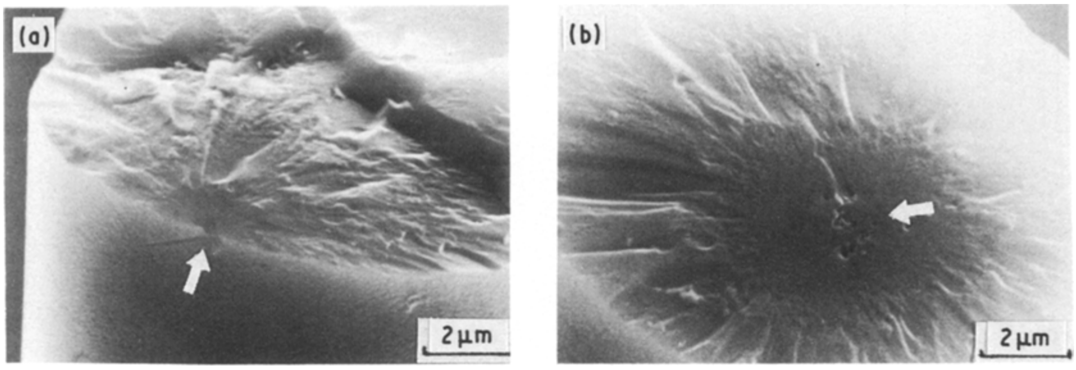

Figure 1 Fractographs of SiC fibres by SEM: (a) fractograph due to a surface defect, (b) fractograph due to an inner defect.

for Carbon Fibres). The fibre diameter was measured using an optical microscope with the micrometer eyepiece. The tensile test was carried out using an Instron-type testing machine (Tensilon, STM-50BP, Toyo Baldwin Co. Ltd) at a strain rate of $0.02\,min^{-1}$. The number of samples varied from 30 to 45 for each type of fibre.

In general, primary fracture surfaces of brittle fibres are often lost because of the fragmentation occurring at the failure, so a test fibre was coated with a jellied dressing pack in order to prevent the primary fracture tips flying away [18]. The pack was so soft that the increment of force caused by its viscosity could be neglected compared with the load necessary to break the fibre itself. After the test, the pack around the tips was removed in water (at about 70° C), and the primary fracture surfaces were observed by a scanning electron microscope (SEM). As for the Al_2O_3 fibres, we restricted our experiment to counting only the kind of the fracture mode in order to apply the results obtained by Nunes [7] to the present analysis.

3.2. Fractographs and Weibull plots

Figs 1 and 2 show typical fractographs of SiC and Al_2O_3 fibres, respectively. It is seen from the figures that SiC fibres are broken in a brittle manner by two kinds of defects, surface and inner defects, as shown by the white arrows of Figs 1a and b. The former defect may be a "flaw" type, but the fibres broken by a "pit" type of defect and undetectable defects on the surfaces were sometimes observed. The later defect is observed to be a "void" type. The fracture surface of Al_2O_3 fibres appears to be granular and it is difficult to detect the initiation point of the failure, as shown in Fig. 2a. Several specimens had a "crooked" section [7], and failed at one of the bent sections as shown macroscopically in Fig. 2b. But the specimens were not always broken at the "crooked" section, some specimens being broken in an uncrooked section as shown in Fig. 2c. The result means that Al_2O_3 fibres have two kinds of fracture mode. Thus, Al_2O_3 fibres show a tendency similar to the result of Nunes [7], so hereafter, his data are applied to the present analysis.

Fig. 3 shows Weibull plots for the tensile strength of SiC fibres. Here the cumulative failure probability is given by a mean rank method [19]. In the figure the fracture surfaces corresponding to all of the plotted points have been observed by SEM at the initiation point of the fracture. The plotted points for fibres which were broken by the inner defect concentrate into the region of high strength, but the points for fibres which were broken by surface defects are scattered to the low-strength side. Here we deal with the "pit" type, "flaw" type and undetected defects as a similar type of a surface defect, because it is difficult to distinguish which type of defect has caused the failure.

Such a characteristics is also exhibited on the plotted points of Al_2O_3 fibres as shown in Fig. 4, on which the Nunes's data for a gauge length of 10 in. (25 mm) (FP-10) have been employed. The points are composed of two groups depending on the kind of the fracture mode, i.e. a group of the points having high strength and a group of the points having low strength labelled C, which means a fibre fractured at a

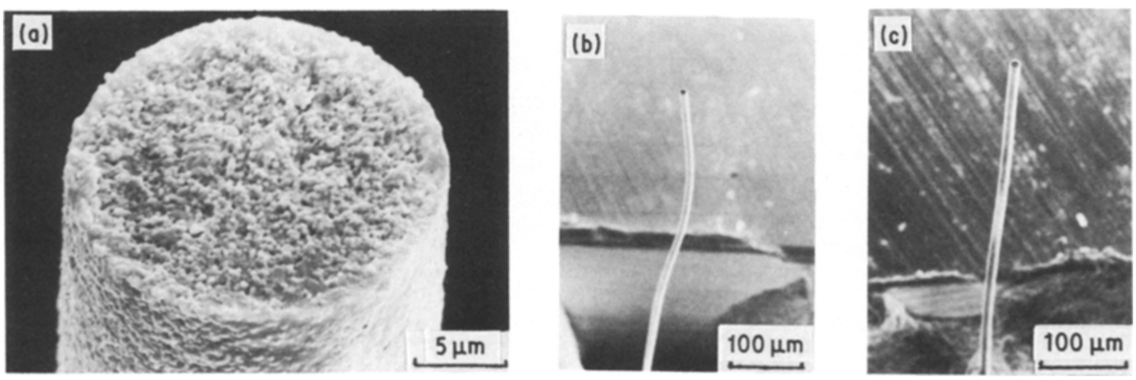

Figure 2 Fractographs of Al_2O_3 fibres: (a) microscopic fractograph, (b) macroscopic fractograph at a crooked section, (c) macroscopic fractograph at an uncrooked section.

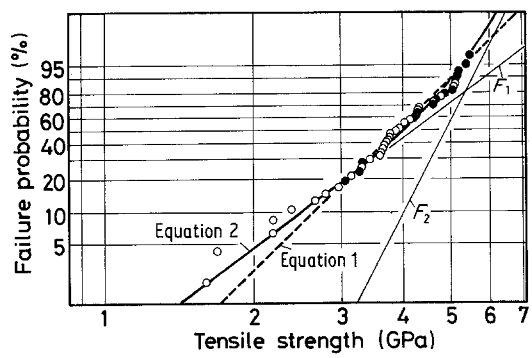

Figure 3 Weibull plots for SiC fibres and the cumulative distribution curves estimated from Equations 1 and 2. (○) Surface defect, (●) inner defect.

"crooked" section. The point positions in the low-strength group are characterized by having more extensive scattering than those in the other groups as shown in both figures.

4. Analytical results and discussion
4.1. Comparison of single and bi-modal Weibull distribution functions

Both SiC and Al_2O_3 fibres have two kinds of fracture mode and the strength distributions are completely changed depending on the fracture mdoe, as mentioned above. For such a distribution, it is doubtful to get a straight line expressed by Equation 1 to fit on to a Weibull probability graph. So, we try to compare the values analysed by Equations 1 and 2, which is our essential purpose. Estimates of the Weibull parameters have been calculated from the data points in Figs 3 and 4. The results are shown in Tables I and II. Maximum logarithm likelihood values obtained from Equation 2 are larger than those from Equation 1 in both cases. Namely, it is proved that the bi-modal Weibull distribution function is a function more closely representing the true distributions.

The cumulative distribution curves calculated from the Weibull estimators in Table I and II have been drawn on to Figs 3 and 4. In Fig. 3, the curve estimated from Equation 2 is in an accurate association with the plotted points. The group of plotted points for each fracture mode appears to be located by competition between two lines, $F_1(\sigma)$ and $F_2(\sigma)$, which mean the strength distributions of the surface and inner defects sub-populations, respectively. On the other hand, the line obtained from Equation 1 does not agree precisely with the points in the region of low strength. The same result has been obtained in Fig. 4. The curve of Equation 2 looks to have been drawn from the results of the competition between $F_1(\sigma)$ and $F_2(\sigma)$, which denote the strength distributions of crooked and uncrooked defect sub-populations, respectively, in Al_2O_3 fibres. The curve also fits more closely to the plotted points than the line of Equation 1. The results show that the bi-modal Weibull distribution is sufficiently applicable to the strength distribution of ceramic fibres, and this may be understood from the concept that a shape parameter is not a constant inherent in a material, but a constant inherent in a defect population.

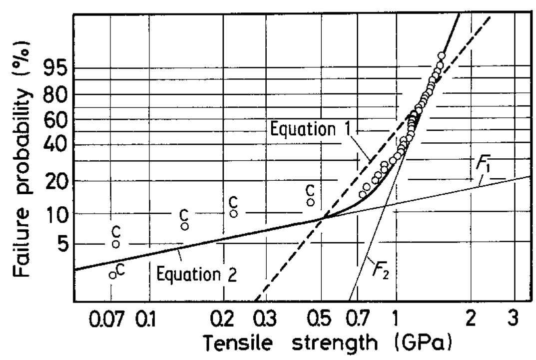

Figure 4 Weibull plots for Al_2O_3 fibre and the cumulative distribution curves estimated from Equations 1 and 2. C = "crooked" fibre.

TABLE I Weibull parameters from Equation 1 and the maximum logarithm likelihood

Fibres	m	σ_0	$\ln L$
SiC	4.70	4.24	−61.75
Al_2O_3	2.95	1.14	−21.97

Beetz [11] has previously proposed a strength distribution for a carbon fibre having two kinds of defect. In his paper, a mixed Weibull distribution function is applied. We consider, however, that this function may be applied to a population composed of two types of fibre. Because it is given from the standpoint of reliability [20], such a case deals with a lifetime distribution of a certain member or device. If the mixed Weibull distribution function is applied to the above situation, the mixed parameters [21] can be easily and conclusively found. It has already been available for the strength analysis of a bundle composed of two types of fibres [22]. On the other hand, the multi-stage procedure method proposed by Pheonix [12] gives an accurate strength distribution curve for the tensile test data of boron fibres. But this method requires much labour for the procedure of preparing the specimens, so in the present paper the analysis by this method has been omitted, and only the results of the constant gauge length tension test have been considered.

4.2. Prediction of strength distribution and the average strength at a different gauge length

Strength distributions should be theoretically predicted at every gauge length. So, the tensile-test of SiC fibres has been additionally carried out at the gauge lengths of 5 and 50 mm without observing by SEM. The cumulative distribution curves of these gauge lengths have been predicted from Equations 6 and 7 on the basis of Weibull parameters estimated at 10 mm gauge length. The results are shown in Fig. 5. It is seen that the curves of Equations 6 and 7 extrapolate to nearly

TABLE II Weibull parameters from Equation 2 and the maximum logarithm likelihood

Fibres	m_1	m_2	σ_{01}	σ_{02}	$\ln L$
SiC	3.64	9.41	4.64	5.08	−57.16
Al_2O_3	0.51	6.52	57.59	1.24	5.25

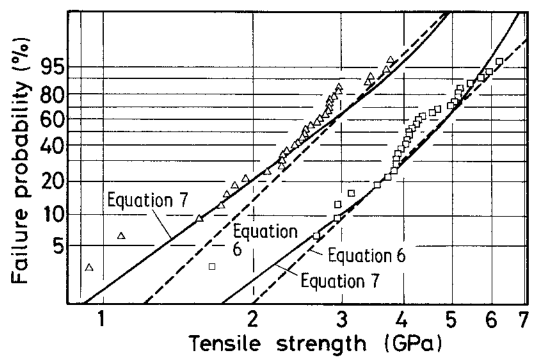

Figure 5 Weibull plots for SiC fibres at gauge lengths of (□) 5 mm and (△) 50 mm, and cumulative distribution curves predicted from Weibull estimates for the gauge length of 10 mm.

the same position for the experimental data for the gauge length of 5 mm. The reason for this may be that the length 5 mm is only one-half of the gauge length used in an estimation of Weibull parameters. For the data of gauge length 50 mm, however, the curve of Equation 7 is fitted more closely than that of Equation 6. Such a tendency is also seen in Al_2O_3 fibres. Fig. 6 shows Weibull plots of data from Nunes [7] at the gauge lengths of 0.5 in. (12.7 mm) (FP-7) and 5 in. (127 mm) (FP-9). The distribution curves are drawn on the figures using Weibull parameters estimated at the gauge length of 10 in. (254 mm). In this case, also, the curves of Equation 7 predict more closely the plots for the gauge length of 0.5 in. and 5 in. than those of Equation 6. Thus, it is proved that the strength distribution for a different gauge length is more accurately predicted from the multi-modal Weibull distribution.

The logarithms of the average strength of SiC and Al_2O_3 fibres are shown by open circles in Figs 7 and 8, plotted against logarithms of the gauge length. The average strength curves of SiC and Al_2O_3 fibres have been calculated from Equations 8 and 9 on the basis of Weibull parameters estimated at gauge lengths of 10 mm and 10 in., respectively. The curves of Equation 9 appear to fit more precisely for each open circle than those of Equation 8. It is proved that the multi-modal Weibull distribution is a more effective distribution to predict the average strength of a different gauge length. Here it is noted that each circle at

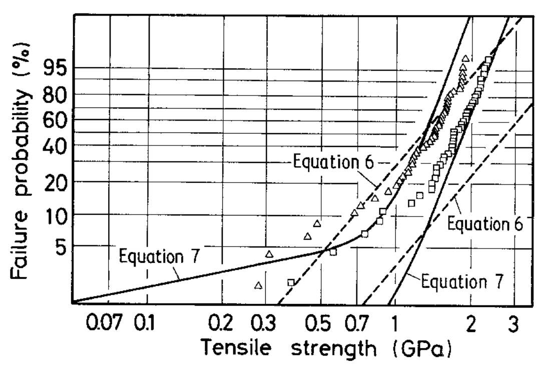

Figure 6 Weibull plots for Al_2O_3 fibres at gauge lengths of (□) 0.5 in. (12.7 mm) and (△) 5 in. (127 mm), and cumulative distribution curves predicted from Weibull estimates for the gauge length of 10 in. (254 mm) (FP-10) [7].

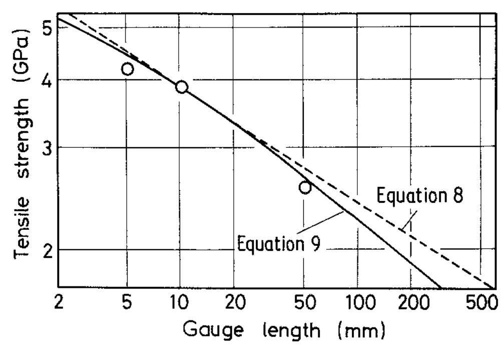

Figure 7 Average strength plots of SiC fibres, and the curves extrapolated from Weibull estimates for the gauge length of 10 mm.

the gauge lengths of 5 mm and 0.5 in. is located at a slightly lower position than the curve of Equation 9. The reason may be that the fractured positions of SiC fibres tested at 5 mm gauge length have sometimes existed in the regions of the fibre clamps, and therefore the average strengths have been underestimated by the clamp effect which has been already pointed out by Pheonix and Sexsmith [23]. This problem can be also solved easily from the concept of the multi-risk model in which the fracture is determined by competition among strength distributions of two clamp regions and a tested region, but in this paper we omit this analysis.

We now resume the discussion concerning a variation of the strength distribution depending on the gauge length, on the basis of a bi-modal Weibull distribution. The strength of reinforcing ceramic fibres is often predicted in the range of aspect ratio from approximately 10 to 100, because in composites a fibre embedded in a matrix exhibits a strength equivalent to an ineffective length (e.g. [24–26]). So, cumulative distribution curves of SiC fibres have been extrapolated from Equation 7 for the gauge lengths of aspect ratio 10 and 100 using Weibull parameters in Table II, as shown in Fig. 9. It is seen that the shorter the gauge length becomes, the larger the curve slopes according to the distribution function F_2. This means a decrease in the scatter of the fibre strength. The reason is that the shape parameter m_1 is smaller than m_2, therefore the distribution line of F_1 moves more to the high-strength side than that of F_2 on a Weibull probability graph. Consequently, the cumulative distribution curve of Equation 7 is progressively governed by F_2. In other words, as the gauge length decreases, one kind of defect

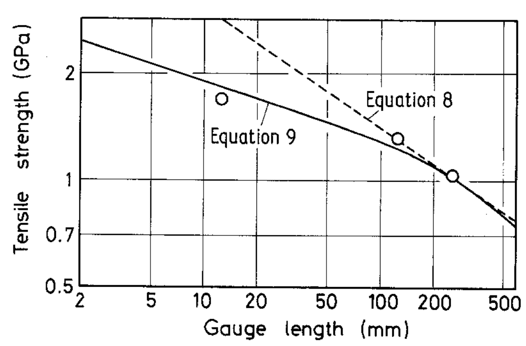

Figure 8 Average strength plots of Al_2O_3 fibres, and the curves extrapolated from Weibull estimates for the gauge length of 10 in. (254 mm) (FP-10) [7].

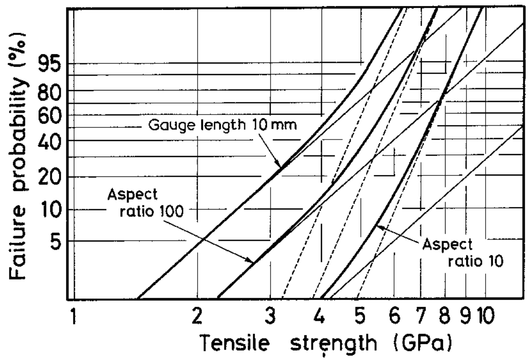

Figure 9 Cumulative distribution curves for SiC fibres predicted from Equation 7 at the aspect ratios of 10 and 100. (——) F_1, (– – –) F_2.

controlling the strength of the fibre translates to the other kind of defect which has a stable strength compared with the former. Because of such a mutual relation between F_1 and F_2, it is assumed that the average strength curve extrapolated from the bi-modal Weibull distribution locates lower in the region of short gauge length than that from the single Weibull distribution. As shown in many reports [7, 10, 27–29], the scatter of the strength of ceramic fibres (for example, as expressed by a coefficient of variation or a shape parameter) has a tendency to decrease with a decrease of gauge length. We consider this is based on the concept mentioned above.

5. Conclusion

For the purpose of evaluating exactly a strength distribution of ceramic fibres, a multi-modal Weibull distribution was applied to the tensile test data of coreless silicon carbide and alumina fibres. This treatment was based on the concept that a fibre fracture was determined by a competition among the strength distributions of defect sub-populations. Since two kinds of fracture mode were observed in both types of fibre by SEM, a bi-modal Weibull distribution was employed in the analysis and compared with the single Weibull distribution usually applied. The results obtained are as follows:

1. The bi-modal Weibull distribution gave a larger value of maximum logarithm likelihood in both types of fibres, and its cumulative distribution curve was in good agreement with the data points, as compared with the curve estimated from a single Weibull distribution.

2. The strength distribution and the average value of a different gauge length were investigated. In this case also it was proved that the values predicted from a bi-modal Weibull distribution corresponded more closely to the data points than those from a single Weibull distribution.

It is finally concluded that the multi-modal Weibull distribution is applicable to the strength distribution of ceramic fibres.

References

1. G. K. LAYDEN, *J. Mater. Sci.* **8** (1973) 1581.
2. J. V. BOGGIO and O. VINGSBO, *ibid.* **11** (1976) 273.
3. J. W. JOHNSON and D. J. THORNE, *Carbon* **7** (1969) 659.
4. B. F. JONES and B. J. S. WILKINS, *Fib. Sci. Tech.* **5** (1972) 315.
5. Z. CHI, T. W. CHOU and G. SHEN, *J. Mater. Sci.* **19** (1984) 3319.
6. G. SIMON and A. R. BUNSELL, *ibid.* **19** (1984) 3649.
7. J. NUNES, AMMRC TR 82-61 (Army Materials and Mechanics Research Center, Watertown, Massachusetts, 1982) p. 1.
8. P. MARTINEAU, M. LAHAYE, R. PAILLER, R. NASLAIN, M. COUZI and F. CREUGE, *J. Mater. Sci.* **19** (1984) 2731.
9. H. FUKUNAGA and K. GODA, in Proceedings of 6th International European Chapter Conference of SAMPE, Scheveningen, May 1985, edited by G. Bartelds and R. J. Schliekelmann (Elsevier, Amsterdam, 1985) p. 125.
10. D. M. COTCHICK, R. C. HINK and R. E. TRESSLER, *J. Compos. Mater.* **9** (1975) 327.
11. C. P. BEETZ Jr, *Fib. Sci. Tech.* **16** (1982) 45.
12. S. L. PHEONIX, ASTM STP 580 (American Society for Testing and Materials, Philadelphia, 1975) p. 77.
13. Y. MATSUO and H. MURATA, *J. Soc. Mater. Sci.* (in Japanese) **34** (1984) 1545.
14. R. J. HERMAN and R. K. N. PATELL, *Technometrics* **13** (1971) 385.
15. T. E. EASLER, R. C. BRADT and R. E. TRESSLER, *J. Amer. Ceram. Soc.* **64** (1981) C-53.
16. K. JAKUS, J. E. RITTER Jr, T. SERVICE and D. SONDERMAN, *ibid.* **64** (1981) C-174.
17. D. SONDERMAN, K. JAKUS, J. E. RITTER Jr, S. YUHASKI Jr and T. H. SERVICE, *J. Mater. Sci.* **20** (1985) 207.
18. J. B. JONES, J. B. BARR and R. SMITH, *ibid.* **15** (1980) 2455.
19. E. J. GUMBEL, "Statics of Extremes" (Japanese translation by T. Kawata, S. Iwai and S. Kase) (Seisan Gijutsu Center Shinsha, Tokyo, 1978) p. 50.
20. N. R. MANN, R. E. SCHAFER and N. D. SINGPURWALLA, "Methods for Statistical Analysis and Life Data" (Wiley, New York, 1974) p. 137.
21. J. H. K. KAO, *Technometrics* **1** (1959) 389.
22. S. L. PHEONIX, *Fib. Sci. Tech.* **7** (1974) 15.
23. S. L. PHEONIX and R. G. SEXSMITH, *J. Compos. Mater.* **6** (1972) 322.
24. B. W. ROSEN, *AIAA Journal* **2** (1964) 1985.
25. C. ZWEBEN, *ibid.* **4** (1968) 2325.
26. K. P. OH, *J. Compos. Mater.* **13** (1979) 311.
27. M. MORITA, H. TAKEDA and I. ARIMA, *J. Jpn. Inst. Met.* (in Japanese) **36** (1972) 1213.
28. P. W. BARRY, *Fib. Sci. Tech.* **11** (1978) 245.
29. S. CHWASTIAK, J. B. BARR and R. DIDCHENKO, *Carbon* **17** (1979) 49.

Received 21 February
and accepted 28 April 1986